Math 404 Special Homework 2 Name:________________________

To receive full credit, you must show all work.

Question 1 This is exactly problem 11 from section 2.2 in the book. Prove that a straight line is the shortest curve that joins two points in $\mathbb{R}^3$. Do this the following way: Let $c : [a, b] \to \mathbb{R}^3$ be an arbitrary curve from $p = c(a)$ to $q = c(b)$. Let $\mathbf{u} = (\mathbf{q} - \mathbf{p})/\|\mathbf{q} - \mathbf{p}\|$.

a) Show that if σ is a straight line segment from p to q, say $\sigma(t) = (1 - t)\mathbf{p} + t\mathbf{q}$, $0 \le t \le 1$, then $L(\sigma) = d(p, q)$.

b) Cauchy-Schwartz implies that $\|c'\| \ge c' \cdot \mathbf{u}$. Use this to deduce that $L(c) \ge d(p, q)$.

c) Show that if $L(c) = d(p, q)$, then c is a straight line segment.

Question 2 Now we are going to investigate the same problem using the calculus of variations. Very often in math or physics, one is interested in minimizing or maximizing a functional. For our purposes a functional F will be a function from some set of functions to $\mathbb{R}$. These are often given by integrals. For example, consider the set $\mathcal{C}$ of all smooth curves c in the plane joining p to q and parametrized on the interval $[a, b]$. Then the length functional L is $L : \mathcal{C} \to \mathbb{R}$ given by

$$L(c) = \int_a^b \|c'\| \, dt$$

If we further assume that c is the graph of a function $y = c(t)$ joining the points $p = (a, c(a))$ to $q = (b, c(b))$, then L can be written as

$$L(c) = \int_a^b \sqrt{1 + (c')^2} \, dt$$

To find the shortest curve joining p to q, we would like to "differentiate L with respect to c" and set the result equal to 0 to find the "critical curves" which we hope are minimums or shortest curves (geodesics).

Here is the general framework in which to do this. Consider a suitably differentiable function $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by $F(t, x, y)$. We wish to find the maxima/minima of the functional

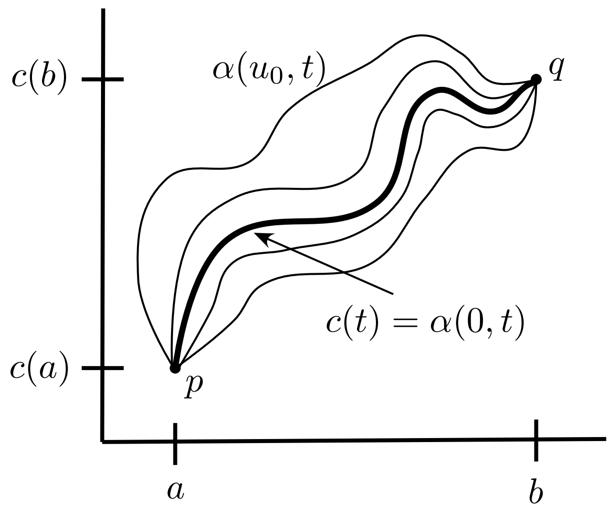
$$J(c) = \int_a^b F(t, c(t), c'(t)) \, dt$$

(To get the length functional, let $F = \sqrt{1 + y^2}$.)

Now we consider a variation of c with endpoints fixed, that is, a function

$$\alpha : (-\varepsilon, \varepsilon) \times [a, b] \to \mathbb{R}$$

such that $\alpha(0, t) = c(t)$ and $\alpha(u, a) = p$ and $\alpha(u, b) = q$ for all $u \in (-\varepsilon, \varepsilon)$. Note that for fixed $u = u_0$, $\alpha(u_0, t)$ is just a curve joining p to q. See the picture. As u varies we get a family of curves which "pass through" c when $u = 0$. Denote the u–th curve by $\overline{\alpha}(u)$.



a) Now it's your turn to do some stuff. For a variation α, show that

$$\frac{d}{du}\left(J(\overline{\alpha}(u))\right)\bigg|_{u=0} = \frac{d}{du}\bigg|_{u=0}\int_a^b F\big(t, \alpha(u,t), \frac{\partial \alpha}{\partial t}(u,t)\big)\, dt$$

$$= \int_a^b \left[\frac{\partial \alpha}{\partial u}(0,t)\frac{\partial F}{\partial x}\big(t, c(t), c'(t)\big) + \frac{\partial^2 \alpha}{\partial u \partial t}(0,t)\frac{\partial F}{\partial y}\big(t, c(t), c'(t)\big)\right] dt$$

Since mixed partials are equal, $\frac{\partial^2 \alpha}{\partial u \partial t} = \frac{\partial^2 \alpha}{\partial t \partial u}$, apply integration by parts to the second term in the integrand and use the fact that endpoints are fixed to conclude

$$\frac{d}{du}\left(J(\overline{\alpha}(u))\right)\bigg|_{u=0} = \int_a^b \frac{\partial \alpha}{\partial u}(0,t)\left[\frac{\partial F}{\partial x}\big(t, c(t), c'(t)\big) - \frac{d}{dt}\left(\frac{\partial F}{\partial y}\big(t, c(t), c'(t)\big)\right)\right] dt$$

b) Thus critical points of J correspond to curves c with

$$\frac{\partial F}{\partial x}\big(t, c(t), c'(t)\big) - \frac{d}{dt}\left(\frac{\partial F}{\partial y}\big(t, c(t), c'(t)\big)\right) = 0$$

This is called the Euler-Lagrange equation of the functional J. Use this to show that straight lines are critical points of the length functional L. ($F(t,x,y) = \sqrt{1+y^2}$.) To show these are actually minima we would have to compute the second derivative of J with respect to u and use the second derivative test. This can be done, but is a big mess!

c) Suppose now that you wanted to find a curve c given as a graph $y = c(t)$ over $[a,b]$, for which the surface of revolution obtained by rotating c about the t–axis has minimal area amongst all curves joining $(a, c(a))$ to $(b, c(b))$. To make the problem interesting we assume that $c(t) > 0$ on $[a,b]$. This will give a so-called minimal surface of revolution.

i) What should the function F be so that the corresponding functional J represents the area of the surface of revolution?

ii) Deduce that a curve c that generates a minimal surface of revolution satisfies the non-linear differential equation

$$1 + \left(\frac{dc}{dt}\right)^2 - c(t)\left(\frac{d^2 c}{dt^2}\right) = 0$$

iii) Miraculously, this differential equation has a closed form solution since the independent variable t is missing. The technique is to let $v = \frac{dc}{dt}$. Then $\frac{d^2 c}{dt^2} = \frac{dv}{dt} = \frac{dv}{dc}\frac{dc}{dt} = v\frac{dv}{dc}$. This converts the given ODE to a separable first-order ODE. Solve it, get another separable ODE, and solve that to find the solution $c(t)$. It turns out that the solution to this differential equation can be rewritten as $c(t) = C \cosh\left(\frac{t+K}{C}\right)$, where C and K are constants determined by initial conditions. The resulting surfaces are called catenoids.